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Limit theorems for tail processes with application to intermediate quantile estimation

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Abstract: A description of the weak and strong limiting behaviour of weighted uniform tail empirical and tail quantile processes is given. The results for the tail quantile process are applied to obtain weak and strong functional limit theorems for a weighted non-uniform tail-quantile-type process based on a random sample from a distribution that satisfies the so called von Mises sufficient condition for being in the domain of max-attraction of a Fréchet distribution. The functional central limit theorem thus obtained yields asymptotic confidence bands for intermediate quantiles.

AMS Subject Classification: Primary 60F05, 60F15; secondary 62G15, 62G30.

Key words and phrases: Asymptotic confidence bands; domain of attraction of a Fréchet distribution; functional central limit theorem; functional law of the iterated logarithm; intermediate quantiles; tail empirical and tail quantile process; weight function.

1. Introduction

Let $U_1, U_2, \ldots$ be a sequence of independent uniform $(0,1)$ random variables and, for each integer $n \geqslant 1$, let

$$F_n(t) = n^{-1} \sum_{i=1}^{n} 1_{\{U_i \leqslant t\}}, \quad -\infty < t < \infty, \tag{1.1}$$

be the empirical distribution function based on the first n of these random variables. Moreover, let

$$Q_n(t) = U_{k,n}, \quad (k-1)/n < t \leqslant k/n, \ k = 1, \ldots, n, \tag{1.2}$$

with $Q_n(0) = U_{1,n}$, where $U_{1,n} \leqslant \cdots \leqslant U_{n,n}$ are the order statistics based on $U_1, \ldots, U_n$, be the empirical quantile function. We write the uniform empirical process as

$$\alpha_n(t) = n^{1/2}(F_n(t) - t), \quad 0 \leqslant t \leqslant 1, \tag{1.3}$$

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the uniform quantile process as

$$\beta_n(t) = n^{1/2}(Q_n(t) - t), \quad 0 \leqslant t \leqslant 1, \tag{1.4}$$

and shall use the notation $\tilde{\beta}_n(t)$ to denote the truncated uniform quantile process, which is equal to $\beta_n(t)$ for $1/(n+1) \leqslant t \leqslant n/(n+1)$ and 0 elsewhere.

Let $\{k_n\}_{n=1}^{\infty}$ denote a sequence of (not necessarily integer) numbers such that for all $n \in \mathbb{N}$,

$$1 \leqslant k_n < n, \quad k_n/n \to 0, \quad k_n \to \infty \ (n \to \infty). \tag{1.5}$$

Now define the tail empirical and tail quantile process in terms of the sequence $\{k_n\}_{n=1}^{\infty}$, by

$$w_n(t) = (n/k_n)^{1/2} \alpha_n(tk_n/n), \quad 0 \leqslant t \leqslant 1; \tag{1.6}$$

$$v_n(t) = (n/k_n)^{1/2} \beta_n(tk_n/n), \quad 0 \leqslant t \leqslant 1. \tag{1.7}$$

The truncated tail quantile process is defined similarly:

$$\tilde{v}_n(t) = (n/k_n)^{1/2} \tilde{\beta}_n(tk_n/n), \quad 0 \leqslant t \leqslant 1. \tag{1.8}$$

The aim of this paper is twofold. Firstly, in Section 2, a description of the weak and strong limiting behaviour of *weighted* versions of w_n and $\tilde{v}_n$ is given. These results are collected from other recent research papers and therefore presented without proof. That section constitutes a survey of results for these tail processes as far as functional central limit theorems (CLT's) and functional laws of the iterated logarithm (LIL's) are concerned. Secondly, in Section 3, weak and strong limit theorems are derived for a weighted non-uniform tail-quantile-type process based on a random sample $X_1, \dots, X_n$ from a distribution F that satisfies the so called von Mises sufficient condition for being in the domain of max-attraction of a Fréchet distribution. The proofs of the results in that section are based on the theorems for $\tilde{v}_n$ in Section 2.

2. Limit theorems for weighted uniform tail processes

For ease of presentation we restrict ourselves in this survey-section to power weight functions, i.e. to functions of the form I^η, $0 \leqslant \eta \leqslant 1$, where I denotes the identity function on $[0,1]$. (Throughout we set the convention $0^0 = 1$, and also $0/0 = 0$.) After the presentation of the results, for the interested reader the papers are cited where the most general versions (w.r.t. weight functions) of the theorems can be found.

To be more precise, in this section we establish limit theorems for the weighted processes w_n/I^η and $\tilde{v}_n/I^\eta$. We consider $\tilde{v}_n/I^\eta$ rather than v_n/I^η since $\sup_{0<t<1/k_n} |v_n(t)|/t^\eta = \infty$ a.s. when $0 < \eta \leqslant 1$. Some more notation is needed. Let $\mathscr{B}$ denote the space of bounded real-valued functions defined on $[0,1]$ with the supremum norm and $\mathscr{S}$ denote the set of absolutely continuous functions $f \in \mathscr{B}$ such that $f(0)=0$ and $\int_0^1 (f'(t))^2\,dt \leqslant 1$, where f' denotes the Lebesgue derivative of f (see Strassen (1964)). For the functional LIL's we need the following stronger condition on the sequence $\{k_n\}_{n=1}^\infty$:

$$k_n \text{ satisfies (1.5)}, \ k_n/n\downarrow, \ k_n\uparrow \text{ and } k_n/l_n \to \infty \ (n\uparrow\infty), \tag{2.1}$$

where here and in the remainder of this paper l_n denotes $\log\log(n\vee 3)$ for $n\geqslant 1$. Finally, it should be noted that we do not explicitly mention the probability spaces involved, but that we assume that they are 'rich enough'.

Theorem 2.1 (CLT for w_n/I^η). *Let $\{k_n\}_{n=1}^\infty$ satisfy* (1.5) *and let $0\leqslant\eta\leqslant 1$. If $\eta<\frac{1}{2}$ then there exists a sequence $\{W_n\}_{n=1}^\infty$ of standard Wiener processes such that*

$$\sup_{0<t\leqslant 1} |w_n(t)-W_n(t)|/t^\eta \xrightarrow{\mathrm{P}} 0 \quad \text{as } n\to\infty. \tag{2.2}$$

Conversely, (2.2) *holding true for some sequence $\{W_n\}_{n=1}^\infty$ of standard Wiener processes, implies $\eta<\frac{1}{2}$.*

The first part of this theorem follows immediately from Csörgő et al. (1986, Corollary 4.2.1). The proof of the converse statement is standard, see also Einmahl (1992).

Theorem 2.2 (LIL for w_n/I^η). *Let $\{k_n\}_{n=1}^\infty$ satisfy* (2.1) *and let $0\leqslant\eta\leqslant 1$.*

(I) *If $\eta=0$ or $0<\eta\leqslant 1$ and $\sum n^{-1} k_n^{(\eta-1/2)/\eta} l_n^{-1/(2\eta)}<\infty$, then almost surely the sequence $\{(2l_n)^{-1/2} w_n/I^\eta\}_{n=1}^\infty$ is relatively compact in $\mathscr{B}$ with set of limit points equal to $\{f/I^\eta\colon f\in\mathscr{S}\}$.*

(II) *If $0<\eta\leqslant 1$ and $\sum n^{-1} k_n^{(\eta-1/2)/\eta} l_n^{-1/(2\eta)}=\infty$, then for any $0<\delta<1$,*

$$\limsup_{n\to\infty} \sup_{0<t\leqslant\delta} (2l_n)^{-1/2}|w_n(t)|/t^\eta = \infty \quad a.s. \tag{2.3}$$

This theorem is due to Einmahl and Mason (1988a) and Mason (1988). More general weight functions are dealt with in Einmahl (1992).

Theorem 2.3 (CLT for $\tilde{v}_n/I^\eta$). *Theorem* 2.1 *holds true verbatim with w_n replaced by $\tilde{v}_n$.*

This result follows from Theorem 2.1 in Csörgő et al. (1986); see for details Csörgő and Horváth (1987, Section 2).

Theorem 2.4 (LIL for $\tilde{v}_n/I^\eta$). *Let $\{k_n\}_{n=1}^\infty$ satisfy* (2.1) *and let $0\leqslant\eta\leqslant 1$.*

(I) *If $k_n^{1-2\eta}/l_n\to\infty$ $(n\to\infty)$, then almost surely the sequence $\{(2l_n)^{-1/2}\tilde{v}_n/I^\eta\}_{n=1}^\infty$ is relatively compact in $\mathscr{B}$ with set of limit points equal to $\{f/I^\eta\colon f\in\mathscr{S}\}$.*

(II) *If $\limsup_{n\to\infty} k_n^{1-2\eta}/l_n<\infty$, then almost surely the sequence $\{(2l_n)^{-1/2}\tilde{v}_n/I^\eta\}_{n=1}^\infty$ fails to be relatively compact in $\mathscr{B}$.*

Finally we present for future reference a theorem which is closely related to Theorem 2.4.

Theorem 2.5. *Let $\{k_n\}_{n=1}^{\infty}$ satisfy (2.1), let $0 \leqslant \eta < \frac{1}{2}$ and assume $k_n^{1-2\eta}/l_n \to \infty$ $(n \to \infty)$. Then we have the existence of a sequence $\{W_n\}_{n=1}^{\infty}$ of independent standard Wiener processes, such that*

$$\sup_{0<t\leqslant 1} \left| \tilde{v}_n(t) - k_n^{-1/2} \sum_{i=1}^{n} W_i(tk_n/n) \right| \Big/ (l_n^{1/2} t^{\eta}) \to 0 \quad a.s. \tag{2.4}$$

Theorems 2.4 and 2.5 are due to Einmahl and Mason (1988b). For functional LIL's for w_n and v_n when $k_n = \mathrm{O}(l_n)$ we refer to Deheuvels and Mason (1990).

3. Limit theorems for a non-uniform tail quantile process

Here we deal with a sequence $X_1, X_2, \ldots$ of independent random variables from a distribution F that satisfies the von Mises sufficient condition for being in the domain of attraction of a Fréchet distribution. For notational convenience we want, in contrast to what is usual in extreme value theory, this condition to hold in the *left* tail of the distribution: throughout this section we assume that F has a positive derivative f in some neighbourhood of $-\infty$ and that for some $\gamma > 0$,

$$\lim_{x \to -\infty} \frac{F(x)}{|x| f(x)} = \gamma. \tag{3.1}$$

Denote, for each integer $n \geqslant 1$, the empirical quantile function by

$$G_n(t) = X_{k,n}, \quad (k-1)/n < t \leqslant k/n, \; k = 1, \ldots, n, \tag{3.2}$$

with $G_n(0) = X_{1,n}$, where $X_{1,n} \leqslant \cdots \leqslant X_{n,n}$ are the order statistics based on $X_1, \ldots, X_n$. Furthermore, the (theoretical) quantile function is defined by

$$G(t) = \inf\{x\colon F(x) \geqslant t\}, \quad 0 < t < 1, \tag{3.3}$$

and the quantile process (in a right neighbourhood of 0) by

$$\varrho_n(t) = n^{1/2} f(G(t))(G_n(t) - G(t)). \tag{3.4}$$

The aforementioned tail-quantile-type process, which is the subject of our study in this section, is (for large n) defined by (cf. 1.7))

$$\begin{aligned} v_n(t) &= \frac{k_n^{1/2} t}{M_n} \left(1 - \frac{G_n(tk_n/n)}{G(tk_n/n)}\right) \\ &= \frac{tk_n/n}{f(G(tk_n/n))\,|G(tk_n/n)|\, M_n} \left(\frac{n}{k_n}\right)^{1/2} \varrho_n\!\left(\frac{tk_n}{n}\right), \quad 0 < t \leqslant 1, \end{aligned} \tag{3.5}$$

where $\{k_n\}_{n=1}^{\infty}$ satisfies (1.5) and where M_n is the left tail version of the well known Hill (1975) estimator (based on $\{[k_n]\}_{n=1}^{\infty}$), i.e.

$$M_n = [k_n]^{-1} \sum_{i=1}^{[k_n]} \log^+ |X_{i,n}| - \log^+ |X_{[k_n]+1,n}|. \tag{3.6}$$

(Here $[x]$ denotes the largest integer $\leqslant x$ and $\log^+ x = \log(x \vee 1)$, for $x \in \mathbb{R}$.) The definition of v_n is motivated by the fact that asymptotic confidence bands for intermediate quantiles $G(tk_n/n)$, $0<t\leqslant 1$, can be obtained, if a functional CLT for v_n is derived. Of course, confidence bands do not follow directly from a CLT for the (general) tail quantile process $(n/k_n)^{1/2}\varrho_n(\cdot\, k_n/n)$, because of the fact that f is unknown.

We are now prepared to present the results of this section, which are weak and strong limit theorems for weighted versions of v_n. Proofs are postponed until the end of this section.

Theorem 3.1. *Let* $\{k_n\}_{n=1}^{\infty}$ *satisfy* (1.5), *let* $0\leqslant\eta<\frac{1}{2}$ *and assume* F *satisfies* (3.1). *Then there exists a sequence* $\{W_n\}_{n=1}^{\infty}$ *of standard Wiener processes such that*

$$\sup_{0<t\leqslant 1} |v_n(t) - W_n(t)|/t^{\eta} \xrightarrow{\mathrm{P}} 0 \quad \text{as } n\to\infty. \tag{3.7}$$

Let $c := c(\alpha, \eta)$ be such that

$$P\Big(\sup_{0<t\leqslant 1} |W_1(t)|/t^{\eta} \geqslant c\Big) = \alpha, \quad 0<\alpha<1, \tag{3.8}$$

and for $x, y \in \mathbb{R}$ define

$$(x,y)^* = \begin{cases} \dfrac{x}{y} & \text{if } x<0 \text{ and } y>0, \\ -\infty & \text{otherwise.} \end{cases} \tag{3.9}$$

Corollary 3.1. *Let* $0<\alpha<1$. *Under the conditions of Theorem* 3.1,

$$P\left(\left(G_n\left(\frac{tk_n}{n}\right), 1-\frac{cM_n}{k_n^{1/2}t^{1-\eta}}\right)^* < G\left(\frac{tk_n}{n}\right) < G_n\left(\frac{tk_n}{n}\right)\Big/\left(1+\frac{cM_n}{k_n^{1/2}t^{1-\eta}}\right), 0<t\leqslant 1\right)$$
$$\to 1-\alpha \quad \text{as } n\to\infty. \tag{3.10}$$

Theorem 3.2. *Let* $\{k_n\}_{n=1}^{\infty}$ *satisfy* (2.1) *and* $(\log k_n)/l_n \to \infty$ $(n\to\infty)$, *let* $0\leqslant\eta<\frac{1}{2}$ *and assume that* F *satisfies* (3.1). *Then there exists a sequence* $\{W_n\}_{n=1}^{\infty}$ *of independent standard Wiener processes, such that*

$$\sup_{0<t\leqslant 1} \left| v_n(t) - k_n^{-1/2} \sum_{i=1}^{n} W_i(tk_n/n) \right| \Big/ (l_n^{1/2} t^{\eta}) \to 0 \quad a.s. \tag{3.11}$$

Corollary 3.2. *Under the assumptions of Theorem* 3.2 *the sequence* $\{(2l_n)^{-1/2}v_n/I^\eta\}_{n=1}^\infty$ *is relatively compact in* $\mathscr{B}$ *with set of limit points equal to* $\{f/I^\eta: f\in\mathscr{S}\}$.

For an easy application of an unweighted version of Theorem 2.1 to estimating the intermediate tail of an arbitrary distribution function F, see Einmahl (1990); Theorem 3.1 can be seen as a partial analogue of that result for intermediate quantiles.

Since the proofs of Theorems 3.1 and 3.2 are very much the same, we only give a detailed proof of Theorem 3.2 (and its corollary). As far as the proof of Theorem 3.1 is concerned we only note that the weak analogue of (3.19) below follows immediately from (3.1) in Csörgő and Horváth (1987) in combination with a slight adaptation of (3.18) in the same paper. (Note that the aforementioned application of our Theorem 2.3 is hidden in the application of their (3.1).)

Proof of Theorem 3.2. Write

$$Y_n(t)=k_n^{-1/2}\sum_{i=1}^n W_i(tk_n/n),\quad 0\leqslant t\leqslant 1, \tag{3.12}$$

$$a_n(t)=\frac{tk_n/n}{f(G(tk_n/n))\,|G(tk_n/n)|\,M_n},\quad 0<t\leqslant 1, \tag{3.13}$$

and observe that

$$\begin{aligned}&\sup_{0<t\leqslant 1}|v_n(t)-Y_n(t)|/(l_n^{1/2}t^\eta)\\&\quad\leqslant \sup_{0<t\leqslant 1}|a_n(t)|\sup_{0<t\leqslant 1}|(n/k_n)^{1/2}\varrho_n(tk_n/n)-Y_n(t)|/(l_n^{1/2}t^\eta)\\&\qquad+\sup_{0<t\leqslant 1}|a_n(t)-1|\sup_{0<t\leqslant 1}|Y_n(t)|/(l_n^{1/2}t^\eta).\end{aligned} \tag{3.14}$$

It is shown in Deheuvels et al. (1988) that under the assumptions of this theorem,

$$M_n\to\gamma\quad\text{a.s.} \tag{3.15}$$

From (3.1) we have

$$\sup_{0<t\leqslant 1}\left|\frac{tk_n/n}{f(G(tk_n/n))\,|G(tk_n/n)|}-\gamma\right|\to 0\quad\text{as } n\to\infty. \tag{3.16}$$

Combining (3.15) and (3.16) yields

$$\sup_{0<t\leqslant 1}|a_n(t)-1|\to 0\quad\text{a.s.} \tag{3.17}$$

From Corollary 4 in Mason (1988) it follows that for $0\leqslant\eta<\frac{1}{2}$,

$$\sup_{0<t\leqslant 1}|Y_n(t)|/(l_n^{1/2}t^\eta)=\mathrm{O}(1)\quad\text{a.s.} \tag{3.18}$$

Combining (3.14), (3.17) and (3.18) we see that, to show (3.11), it remains to prove that

$$\sup_{0<t\leqslant 1} |(n/k_n)^{1/2}\varrho_n(tk_n/n)^{1/2} - Y_n(t)|/(l_n^{1/2}t^{\eta}) \to 0 \quad \text{a.s.} \tag{3.19}$$

Recall the definition of v_n in (1.7) and assume without loss of generality that $X_i = G(U_i)$, $i \in \mathbb{N}$. Now the mean value theorem yields

$$(n/k_n)^{1/2}\varrho_n(tk_n/n) = (f(G(tk_n/n))/f(G(\xi_n(t))))\,v_n(t), \quad 0<t\leqslant 1,$$

with $(tk_n/n)\wedge Q_n(tk_n/n) \leqslant \xi_n(t) \leqslant (tk_n/n)\vee Q_n(tk_n/n)$. Theorem 4 in Wellner (1978) implies that for $0<\varepsilon<1$ arbitrary,

$$\sup_{\varepsilon\leqslant t\leqslant 1}\left|\frac{Q_n(tk_n/n)}{tk_n/n} - 1\right| \to 0 \quad \text{a.s.}$$

Hence, since $f(G(t)) = t^{\gamma+1}L(t)$ in a right neighbourhood of 0, where L is slowly varying at 0 (see Csörgő and Horváth (1987) and Horváth (1987)), we have that

$$\sup_{\varepsilon\leqslant t\leqslant 1}\left|\frac{f(G(tk_n/n))}{f(G(\xi_n(t)))} - 1\right| \to 0 \quad \text{a.s.}$$

This in combination with Theorems 2.4 and 2.5 implies

$$\sup_{\varepsilon\leqslant t\leqslant 1} |(n/k_n)^{1/2}\varrho_n(tk_n/n) - Y_n(t)|/(l_n^{1/2}t^{\eta}) \to 0 \quad \text{a.s.} \tag{3.20}$$

We obtain from Theorem 5 in Wellner (1978),

$$\sup_{l_n/k_n\leqslant t\leqslant 1}\left(\frac{\xi_n(t)}{tk_n/n}\vee\frac{tk_n/n}{\xi_n(t)}\right) = \mathrm{O}(1) \quad \text{a.s.},$$

which easily implies

$$\sup_{l_n/k_n\leqslant t\leqslant 1} f(G(tk_n/n))/f(G(\xi_n(t))) = \mathrm{O}(1) \quad \text{a.s.} \tag{3.21}$$

Furthermore, we need Lemma 7 in Mason (1988):

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\sup_{0<t\leqslant\varepsilon} |Y_n(t)|/(l_n^{1/2}t^{\eta}) = 0 \quad \text{a.s.} \tag{3.22}$$

Combining (3.21), Theorem 2.5 and (3.22) yields

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\sup_{l_n/k_n\leqslant t\leqslant\varepsilon} (n/k_n)^{1/2}|\varrho_n(tk_n/n)|/(l_n^{1/2}t^{\eta}) = 0 \quad \text{a.s.} \tag{3.23}$$

From (3.20), (3.22) and (3.23) we see that it remains to show that

$$\sup_{0<t\leqslant l_n/k_n} (n/k_n)^{1/2}|\varrho_n(tk_n/n)|/(l_n^{1/2}t^{\eta}) \to 0 \quad \text{a.s.}$$

First consider

$$z_n := \sup_{n/(k_n(n+1))\leqslant t\leqslant l_n/k_n} (n/k_n)^{1/2}|\varrho_n(tk_n/n)|/(l_n^{1/2}t^{\eta}). \tag{3.24}$$

Theorem 6 in Wellner (1978) implies that for any $\varepsilon>0$,

$$\sup_{n/(k_n(n+1))\leqslant t\leqslant l_n/k_n} f(G(tk_n/n))/f(G(\xi_n(t)))$$
$$= \mathrm{O}((\log n)^{(\gamma+1)(1+\varepsilon)}) \quad \text{a.s.} \tag{3.25}$$

Moreover from Theorem 3 in Einmahl and Mason (1988b) it easily follows that

$$\sup_{n/(k_n(n+1))\leqslant t\leqslant l_n/k_n} |v_n(t)|/(l_n^{1/2}t^\eta) = \mathrm{O}(l_n^{1/2}/k_n^{1/2-\eta}) \quad \text{a.s.} \tag{3.26}$$

Combining (3.24)–(3.26) and the fact that $(\log k_n)/l_n\to\infty$ yields that $z_n\to 0$ a.s. The proof of

$$\sup_{0<t\leqslant n/(k_n(n+1))} (n/k_n)^{1/2}|\varrho_n(tk_n/n)|/(l_n^{1/2}t^\eta)\to 0 \quad \text{a.s.}$$

follows from an adaptation of the proof of (3.18) in Csörgő and Horváth (1987). The main idea there is to write, for $0<t\leqslant n/(k_n(n+1))$,

$$G_n\left(\frac{tk_n}{n}\right)-G\left(\frac{tk_n}{n}\right)$$
$$=X_{1,n}-G\left(\frac{tk_n}{n}\right)$$
$$=\left\{G_n\left(\frac{1}{n+1}\right)-G\left(\frac{1}{n+1}\right)\right\}+\left\{G\left(\frac{1}{n+1}\right)-G\left(\frac{tk_n}{n}\right)\right\}.$$

For brevity's sake, the straightforward computation following thereafter is omitted. □

Proof of Corollary 3.2. Combination of Theorem 2 and Corollary 4 in Mason (1988) with Theorem 3.2 proves this corollary. □

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